

On the \mathbb{Z}_q -MacDonald Code and its Weight Distribution of dimension 3

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Abstract: In this paper, we determine the parameters of \mathbb{Z}_q -MacDonald Code of dimension k for any positive integer $q \geq 2$. Further, we have obtained the weight distribution of \mathbb{Z}_q -MacDonald code of dimension 3 and furthermore, we have given the weight distribution of \mathbb{Z}_q -Simplex code of dimension 3 for any positive integer $q \geq 2$.

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1 Introduction

A code C is a subset of \mathbb{Z}_q^n where \mathbb{Z}_q is the set of all integers modulo q and n is any positive integer. Let $x, y \in \mathbb{Z}_q^n$. Then the *Hamming distance* between x and y is the number of

coordinates in which they differ. It is denoted by $d(x, y)$. Vividly $d(x, y) = wt(x - y)$, the number of non-zero coordinates in $x - y$ is called the *Hamming weight* of $x - y$. The *minimum Hamming distance* d of C is defined as

$$d = \min\{d(x, y) \mid x, y \in C \text{ and } x \neq y\} = \min\{wt(x - y) \mid x, y \in C \text{ and } x \neq y\}$$

and the *minimum Hamming weight* of C is $\min\{wt(c) \mid c \in C \text{ and } c \neq 0\}$. Hereafter we simply call the minimum Hamming distance and the minimum Hamming weight, the minimum distance and the minimum weight respectively. A code over \mathbb{Z}_q of length n , cardinality M with the minimum distance d is called an (n, M, d) \mathbb{Z}_q -code. Let C be an (n, M, d) \mathbb{Z}_q -code. For $0 \leq i \leq n$, let A_i be the number of codewords of the Hamming weight i . Then $\{A_i\}_{i=0}^n$ is called the *weight distribution* of the code C .

We know pretty well that \mathbb{Z}_q is a group under the addition modulo q . Then \mathbb{Z}_q^n is a group under coordinate-wise addition modulo q . C is said to be a \mathbb{Z}_q -linear code if C is a subgroup of \mathbb{Z}_q^n . In fact, it is a free \mathbb{Z}_q -module. Since \mathbb{Z}_q^n is a free \mathbb{Z}_q -module, it has a basis. Therefore, every \mathbb{Z}_q -linear code has a basis. Since \mathbb{Z}_q^n has a finite basis, \mathbb{Z}_q -linear code has a finite dimension. Since \mathbb{Z}_q^n is finitely generated \mathbb{Z}_q -module, it implies that C is a finitely generated submodule of \mathbb{Z}_q^n . The cardinality of a minimal generating set of C is called the rank of the code C [15]. A generator matrix of C is a matrix the rows of which generate C . Any linear code C over \mathbb{Z}_q with generator matrix G is permutation-equivalent to a code with generator matrix of the form

$$\begin{bmatrix} I_k & A_{01} & A_{02} & \cdots & A_{0s-1} & A_{0s} \\ 0 & z_1 I_{k_1} & z_1 A_{12} & \cdots & z_1 A_{1s-1} & z_1 A_{1s} \\ 0 & 0 & z_2 I_{k_2} & \cdots & z_2 A_{2s-1} & z_2 A_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_{s-1} I_{k_{s-1}} & z_{s-1} A_{s-1s} \end{bmatrix}$$

where A_{ij} are matrices over \mathbb{Z}_q , $\{z_1, z_2, \dots, z_{s-1}\}$ are the zero-divisors in \mathbb{Z}_q and the columns are grouped into blocks of sizes k, k_1, \dots, k_{s-1} respectively. Then $|C| = q^k (\frac{q}{z_1})^{k_1} (\frac{q}{z_2})^{k_2} \dots (\frac{q}{z_{s-1}})^{k_{s-1}}$. If $k_1 = k_2 = \dots = k_{s-1} = 0$, then the code C is called k -dimensional code. Every k dimension \mathbb{Z}_q -linear code with length n and the minimum distance d is called an $[n, k, d]$ \mathbb{Z}_q -linear code.

There are many researchers doing research on codes over finite rings [1], [4], [8], [13] and [16]. In the last decade, there have been many number of researchers doing research on codes over \mathbb{Z}_4 and \mathbb{Z}_q [3], [7], [9], [10] and [14]. Further, in [11], they have determined the

parameters of \mathbb{Z}_q -Simplex codes of dimension k and in [12], they have obtained the weight distribution of \mathbb{Z}_q -Simplex codes of dimension 2 for any positive integer $q \geq 2$.

Theorem 1.1. [11] *The \mathbb{Z}_q -Simplex code of dimension k is an $[n_k = \frac{q^k-1}{q-1}, k, d_k = \frac{q}{p}(p-1)n_{k-1} + 1]$ \mathbb{Z}_q -linear code where $p > 1$ is the smallest divisor of q .*

In [5], they have defined a \mathbb{Z}_q -linear code which is similar to the MacDonald code over finite field. But it gives different weight distribution. In the generator matrix $G_k(q)$ of \mathbb{Z}_q -Simplex code $S_k(q)$ of dimension k , by deleting the matrix

$$\begin{bmatrix} O \\ G_u(q) \end{bmatrix}$$

where $2 \leq u \leq k-1$ and O is $(k-u) \times \frac{q^u-1}{q-1}$ zero matrix, they have obtained

$$G_{k,u}(q) = \left(G_k(q) \setminus \begin{pmatrix} 0 \\ G_u(q) \end{pmatrix} \right) \quad (1.1)$$

for $2 \leq u \leq k-1$ and $(A \setminus B)$ is a matrix obtained from the matrix A by removing the matrix B . A code generated by the matrix $G_{k,u}(q)$ is called \mathbb{Z}_q -MacDonald code. It is denoted by $M_{k,u}(q)$. It is clear that the dimension of this code is k . The Quaternary MacDonald codes were discussed in [6] and the MacDonald codes over finite field were discussed in [2].

In this correspondence, we concentrate on \mathbb{Z}_q -MacDonald Code. In Section 2, we determine the parameters of \mathbb{Z}_q -MacDonald code of dimension k and in Section 3, we obtain the weight distribution of \mathbb{Z}_q -MacDonald code of dimension 3, for any positive integer $q \geq 2$. In Section 4, we find the weight distribution of \mathbb{Z}_q -Simplex code of dimension 3, for any positive integer $q \geq 2$.

2 Minimum Distance of \mathbb{Z}_q -MacDonald Code of dimension k

In Equation 1.1, if we put $u = k-1$, then a generator matrix of k -dimensional \mathbb{Z}_q -MacDonald code is

$$G_{k,k-1}(q) = \left[\begin{array}{c|c|c|c|c} 1 & 11 \dots 1 & 22 \dots 2 & \dots & q-1 \ q-1 \dots q-1 \\ \hline 0 & & & & \\ \vdots & G_{k-1}(q) & G_{k-1}(q) & \dots & G_{k-1}(q) \\ \hline 0 & & & & \end{array} \right]$$

where $G_{k-1}(q)$ is a generator matrix of \mathbb{Z}_q -Simplex code of dimension $k-1$. Then this matrix generates the code

$$M_{k,k-1}(q) = \{(0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1}) \mid \alpha \in \mathbb{Z}_q, c \in S_{k-1}(q)\}$$

where $\mathbf{i} = ii \cdots i \in \mathbb{Z}_q^n$ and $n = \frac{q^{k-1}-1}{q-1} = n_{k-1}$. The code generated by the Matrix $G_{k,k-1}(q)$ is a $[q^{k-1}, k, d(M_{k,k-1}(q))]$ \mathbb{Z}_q -linear code.

Case (i). Let $\alpha = 0$. Then

$$\min\{wt(0cc \cdots c) \mid c \in S_{k-1}(q)\} = (q-1)d(S_{k-1}(q)) = (q-1)\left(\frac{q}{p}(p-1)n_{k-2} + 1\right) \quad (2.1)$$

where $p > 1$ is the smallest divisor of q .

Case (ii). Let $\alpha \neq 0$.

Subcase (i). Let $\alpha \in \mathbb{Z}_q$ with $(\alpha, q) = 1$. If $\alpha i = \alpha j$, then $\alpha(i-j) = 0$. Since α is a unit, it implies $i = j$. Therefore $\{\alpha\mathbf{1}, \alpha\mathbf{2}, \dots, \alpha(\mathbf{q} - \mathbf{1})\} = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{q} - \mathbf{1}\}$.

Consider

$$\begin{aligned} wt((0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1})) &= wt((0cc \cdots c) + (\alpha \alpha\mathbf{1} \alpha\mathbf{2} \cdots \alpha(\mathbf{q} - \mathbf{1}))) \\ &= wt((0cc \cdots c) + (\alpha \mathbf{1} \mathbf{2} \cdots \mathbf{q} - \mathbf{1})) \\ &= 1 + \sum_{i=1}^{q-1} wt(c + \mathbf{i}) \\ wt((0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1})) &= 1 + \sum_{i=1}^{q-1} wt(-c + \mathbf{i}) \text{ since } S_{k-1}(q) \text{ is } \mathbb{Z}_q\text{-linear.} \end{aligned}$$

Let $n(i)$ be the number of i coordinates in $c \in S_{k-1}(q)$ where $i = 0, 1, 2, \dots, q-1$. Then for $0 \leq i \leq q-1$, $wt(-c + \mathbf{i}) = n - n(i)$, where n is the length of $S_{k-1}(q)$. Therefore,

$$\begin{aligned} wt((0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1})) &= 1 + \sum_{i=1}^{q-1} (n - n(i)) \\ &= 1 + (q-1)n - \sum_{i=1}^{q-1} n(i) \\ &= 1 + (q-1)n - (n - n(0)) \\ wt((0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1})) &= 1 + (q-2)n + n(0) \text{ for all } c \in S_{k-1}(q). \end{aligned}$$

Therefore,

$$\min \left\{ wt((0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1})) \mid c \in S_{k-1}(q), \alpha \in U(\mathbb{Z}_q) \right\} = 1 + (q-2)n + \min_{c \in S_{k-1}(q)} \{n(0)\},$$

where $U(\mathbb{Z}_q)$ is the set of all units in \mathbb{Z}_q . Thus, we have

$$\min \{ wt((0cc \cdots c) + \alpha(1\mathbf{1}\mathbf{2} \cdots \mathbf{q} - \mathbf{1})) \mid (\alpha, q) = 1 \} = 1 + (q-2)n + \min_{c \in S_{k-1}(q)} \{n(0)\} \quad (2.2)$$

The largest weight codeword of $S_{k-1}(q)$ gives the minimum value of the above Equation 2.2.

Subcase (ii). Let $(\alpha, q) \neq 1$ and $o(\alpha) = d$. Then, $\{\alpha 1, \alpha 2, \dots, \alpha(q-1)\} = \{\alpha 1, \alpha 2, \dots, \alpha(d-1), 0\}$. Clearly, in $\{\alpha 1, \alpha 2, \dots, \alpha(q-1)\}$, each non-zero αi appears $\frac{q}{d}$ times and zero appears $(\frac{q}{d} - 1)$ times.

Consider

$$\begin{aligned} wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) &= wt((0cc \dots c) + (\alpha \alpha 1 \alpha 2 \dots \alpha(\mathbf{q} - 1))) \\ wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) &= 1 + \frac{q}{d} \left\{ wt(\alpha 1 + c) + wt(\alpha 2 + c) + \dots + wt(\alpha(\mathbf{d} - 1) + c) \right\} \\ &\quad + \left(\frac{q}{d} - 1 \right) wt(c) \end{aligned} \quad (2.3)$$

If $c_i \notin \langle \alpha \rangle$ for all i , then $wt(\alpha i + c) = n$. Therefore, Equation 2.3 becomes

$$wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) = 1 + \frac{q}{d} (d-1)n + \left(\frac{q}{d} - 1 \right) wt(c) \quad (2.4)$$

If $c_i \in \langle \alpha \rangle$ for a few r c_i 's, Equation 2.3 becomes

$$\begin{aligned} wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) &= 1 + \frac{q}{d} \left[(d-1-r)n + (n - n(c_1)) + (n - n(c_2)) + \dots + \right. \\ &\quad \left. (n - n(c_r)) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \\ wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) &= 1 + \frac{q}{d} \left[(d-1)n - \sum_{i=1}^r n(c_i) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \end{aligned} \quad (2.5)$$

If there is a $c \in S_{k-1}(q)$ such that $c_i \in \langle \alpha \rangle$ for all i , then $\sum_{i=1}^d n(c_i) = n$. Therefore, Equation 2.3 becomes

$$\begin{aligned} wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) &= 1 + \frac{q}{d} \left[(n - n(c_1)) + (n - n(c_2)) + \dots + \right. \\ &\quad \left. (n - n(c_{d-1})) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \\ &= 1 + \frac{q}{d} \left[(n - n(\alpha)) + (n - n(2\alpha)) + \dots + \right. \\ &\quad \left. (n - n((d-1)\alpha)) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \\ wt((0cc \dots c) + \alpha(112 \dots \mathbf{q} - 1)) &= 1 + \frac{q}{d} \left[(d-1)n - \sum_{i=1}^{d-1} n(i\alpha) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \end{aligned}$$

Since $n = \sum_{i=1}^{d-1} n(c_i) + n(0)$, we get

$$\begin{aligned}
wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) &= 1 + \frac{q}{d} \left[(d-1)n - (n - n(0)) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \\
&= 1 + \frac{q}{d} \left[(d-2)n + n(0) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \\
&= 1 + \frac{q}{d} \left[(d-2)n + n(0) \right] + \left(\frac{q}{d} - 1 \right) (n - n(0)), \\
&\quad \text{where } c_i \in \langle \alpha \rangle \\
&= 1 + \frac{q}{d} (d-2)n + \frac{q}{d} n(0) + \frac{q}{d} n - n - \frac{q}{d} n(0) + n(0) \\
wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) &= 1 + (q-1)n - \frac{q}{d} n + n(0) \tag{2.6}
\end{aligned}$$

From Equations 2.4, 2.5 and 2.6, we have

$$1 + \frac{q}{d} \left[(d-2)n + n(0) \right] + \left(\frac{q}{d} - 1 \right) wt(c) \leq wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) \leq 1 + \frac{q}{d} \left[(d-1)n \right] + \left(\frac{q}{d} - 1 \right) wt(c)$$

for all $c \in S_{k-1}(q)$ and $(\alpha, q) \neq 1$.

If there exists $c \in S_{k-1}(q)$ such that $c_i \in \langle \alpha \rangle$ and d is smaller, then the Equation 2.6 gives the smaller value. That is,

$$\min_{c \in S_{k-1}(q)} \left\{ wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) \mid (\alpha, q) \neq 1 \right\} = 1 + \left(q - \frac{q}{d} - 1 \right) n + \left\{ \min_{c \in S_{k-1}(q)} n(0) \right\} \tag{2.7}$$

where $n(0)$ is the number of zeros in c such that $c_i \in \langle \alpha \rangle$ and α must be smaller order element. Therefore,

$$\begin{aligned}
\min \left\{ wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) \mid \alpha \in \mathbb{Z}_q, c \in S_{k-1}(q) \right\} &= \min \left\{ (q-1)d(S_{k-1}(q)), 1 + (q-2)n + \right. \\
&\quad \left. \min_{c \in S_{k-1}(q)} \{n(0)\}, 1 + \left(q - \frac{q}{d} - 1 \right) n + \right. \\
&\quad \left. \min_{c \in S_{k-1}(q)} n(0) \right\} \\
&= \min \left\{ (q-1)d(S_{k-1}(q)), \right. \\
&\quad \left. 1 + \left(q - \frac{q}{d} - 1 \right) n + \min_{c \in S_{k-1}(q)} n(0) \right\}
\end{aligned}$$

Let α be a least order non-zero element in \mathbb{Z}_q . Since $011 \cdots 1 \in S_2(q)$, it implies that $c = 0\alpha\alpha \cdots \alpha \in S_2(q)$. Therefore, for $k = 3$, the above Equation 2.7 becomes

$$\min_{c \in S_2(q)} \{ wt(0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1) \} = 1 + \left(q - \frac{q}{d} - 1 \right) n_2 + n(0)$$

Since $n(0) = 1$, it implies that,

$$\min_{c \in S_2(q)} \{ wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) \} = 2 + \left(q - \frac{q}{d} - 1 \right) n_2$$

For $k = 3$, Equation 2.1 gives $\min_{c \in S_2(q)} \{wt(0cc \cdots c)\} = (q-1)d(S_2(q))$ and Equation 2.2 gives $\min_{c \in S_2(q)} \{wt(0cc \cdots c) + \alpha(\mathbf{112} \cdots \mathbf{q} - \mathbf{1})\} = 1 + (q-2)n_2 + 1 = 2 + (q-2)n_2$. Therefore, the minimum distance of $M_{3,2}(q)$ is

$$d(M_{3,2}(q)) = \min \left\{ (q-1)d(S_2(q)), 2 + \left(q - \frac{q}{d} - 1\right)n_2 \right\}.$$

Since $d(S_2(q)) = \frac{q}{p}(p-1) + 1$, it follows that $d(M_{3,2}(q)) = 2 + \left(q - \frac{q}{d} - 1\right)n_2$.

For $k = 4$, in $S_3(q)$, the codeword $c = 0\alpha\alpha \cdots \alpha \in S_2(q)$ is repeated q times and $c' = c0c \cdots c$ is a codeword in $S_3(q)$ which gives the minimum number of zeros, and all coordinates of c' are in $\langle \alpha \rangle$. The number of zeros in c' is $q+1$. Hence, Equation 2.7 becomes

$$\min_{c \in S_3(q)} \{wt((0cc \cdots c) + \alpha(\mathbf{112} \cdots \mathbf{q} - \mathbf{1}))\} = 1 + \left(q - \frac{q}{d} - 1\right)n_3 + (q+1).$$

For $k = 5$, the codeword $c' \in S_3(q)$ is repeated q times in $S_4(q)$ and hence $c'' = c'0c' \cdots c'$ is codeword in $S_4(q)$ which gives the minimum number of zeros, and its coordinates are in $\langle \alpha \rangle$. The number of zeros in c'' is $[q(q+1)] + 1$. Hence, Equation 2.7 becomes

$$\min_{c \in S_4(q)} \{wt((0cc \cdots c) + \alpha(\mathbf{112} \cdots \mathbf{q} - \mathbf{1}))\} = 1 + \left(q - \frac{q}{d} - 1\right)n_4 + [q(q+1) + 1].$$

In general, for any k , in $S_{k-1}(q)$, there is a codeword $c \in S_{k-2}(q)$ the coordinates of which are in $\langle \alpha \rangle$ with minimum number of zeros $\frac{q^{k-3}-1}{q-1}$ and hence $c_1 = c0c \cdots c$ is a codeword in $S_{k-1}(q)$ which gives the minimum number of zeros, and its coordinates are in $\langle \alpha \rangle$. The number of zeros in c_1 is $\frac{q^{k-2}-1}{q-1}$. Hence, Equation 2.7 becomes

$$\min_{c_1 \in S_{k-1}(q)} \{wt((0c_1c_1 \cdots c_1) + \alpha(\mathbf{112} \cdots \mathbf{q} - \mathbf{1}))\} = 1 + \left(q - \frac{q}{d} - 1\right)n_{k-1} + \frac{q^{k-2}-1}{q-1}.$$

Therefore,

$$d(M_{k,k-1}(q)) = 1 + \left(q - \frac{q}{d} - 1\right)n_{k-1} + \frac{q^{k-2}-1}{q-1}.$$

Thus, we have

Theorem 2.1. *The \mathbb{Z}_q -MacDonald code $M_{k,k-1}(q)$ is a $[q^{k-1}, k, 1 + (q - \frac{q}{d} - 1)(\frac{q^{k-1}-1}{q-1}) + \frac{q^{k-2}-1}{q-1}]$ \mathbb{Z}_q -linear code where $d > 1$ is the smallest divisor of q .*

3 Weight Distribution of \mathbb{Z}_q -MacDonald Code of Dimension 3

Let

$$G_{3,2}(q) = \left[\begin{array}{c|cccccc|cccccc|c|cccccc} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 & 2 & 2 & 2 & \cdots & 2 & \cdots & q-1 & q-1 & q-1 & q-1 & \cdots & q-1 \\ \hline 0 & 0 & 1 & 1 & 2 & \cdots & q-1 & 0 & 1 & 1 & 2 & \cdots & q-1 & \cdots & 0 & 1 & 1 & 2 & \cdots & q-1 \\ \hline 0 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & \cdots & 1 & 0 & 1 & 1 & 1 & \cdots & 1 \end{array} \right]$$

Then by Theorem 2.1, this matrix generates $[q^2, 3, 2+(q-\frac{q}{d}-1)(q+1)]$ \mathbb{Z}_q -linear code where $d > 1$ is the smallest divisor of q . It is $M_{3,2}(q) = \{(0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1) \mid \alpha \in \mathbb{Z}_q\}$.

In [12], they have given the weight distribution of 2-dimensional \mathbb{Z}_q -Simplex code as

Theorem 3.1. [12] For any integer $q \geq 2$, the weight distribution of \mathbb{Z}_q -Simplex code of dimension 2 is

$$\begin{aligned} A_0 &= 1 \\ A_q &= q\phi(q) + q - 1 \\ A_{q-\frac{q}{d}+1} &= d\phi(d), \text{ for } d|q \text{ and } d \neq 1, d \neq q \\ A_{q+1} &= q(q-1) - \sum_{d|q, d \neq 1} d\phi(d). \end{aligned}$$

where $d > 1$ is the smallest divisor of q .

Note that there is only one codeword in $S_2(q)$ such that $n(0) = n$, $d\phi(d)$ codewords in $S_2(q)$ such that $n(0) = \frac{q}{d}$, for all $d|q$, $d \neq 1$ and $d \neq q$, $q\phi(q) + q - 1$ codewords in $S_2(q)$ such that $n(0) = 1$ and $q(q-1) - \sum_{d|q, d \neq 1} d\phi(d)$ codewords in $S_2(q)$ such that $n(0) = 0$.

Now, we consider the code $M_{3,2}(q)$.

Case (i). If $\alpha = 0$, then

$$\begin{aligned} wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) &= wt(0cc \cdots c) \\ &= (q-1)wt(c), \text{ for all } c \in S_2(q) \end{aligned} \quad (3.1)$$

By Theorem 3.1, we get the following weights:

$$\left\{ \begin{array}{l} \text{Number of zero weight codeword is 1.} \\ \text{Number of } (q-1)(q-\frac{q}{d}+1) \text{ weight codeword is } d\phi(d) \text{ where } d|q, d \neq 1 \text{ and } d \neq q. \\ \text{Number of } (q-1)q \text{ weight codeword is } q\phi(q) + q - 1. \\ \text{Number of } (q-1)(q+1) = q^2 - 1 \text{ weight codeword is } q(q-1) - \sum_{d|q, d \neq 1} d\phi(d). \end{array} \right. \quad (3.2)$$

Case (ii). If $(\alpha, q) = 1$, then by Equation 2.2,

$$wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) = 1 + (q - 2)n + n(0), \quad (3.3)$$

for all $c \in S_2(q)$, and $n(0)$ is the number of zeros in c . By Theorem 3.1, in $M_{3,2}(q)$,

$$\begin{cases} \text{Number of } 1 + (q - 1)n \text{ weight codeword is } 1.\phi(q). \\ \text{Number of } 1 + (q - 2)n + \frac{q}{d} \text{ weight codeword is } (d\phi(d)).\phi(q), \text{ for all } d|q, d \neq 1 \text{ and } d \neq q. \\ \text{Number of } 2 + (q - 2)n \text{ weight codeword is } (q\phi(q) + q - 1).\phi(q). \\ \text{Number of } 1 + (q - 2)n \text{ weight codeword is } (q(q - 1) - \sum_{d|q, d \neq 1} d\phi(d)).\phi(q). \end{cases} \quad (3.4)$$

Case (iii). If α is not relatively prime to q , then by Equation 2.6, we have

$$wt((0cc \cdots c) + \alpha(112 \cdots \mathbf{q} - 1)) = 1 + (q - 1)n - \frac{q}{d} \sum_{i=0}^{d-1} n(\mathbf{i}\alpha) + n(0) \quad (3.5)$$

where $n(\mathbf{i}\alpha)$ is the number of $\mathbf{i}\alpha$'s in c .

If we know the details of coordinates in c , we can get the remaining weights of $M_{3,2}(q)$.

Example 3.2. For $q = 4, k = 3$, the Matrix

$$G_{3,2}(4) = \left[\begin{array}{c|c|c|c} 1 & 1111 & 2222 & 3333 \\ \hline 0 & 01123 & 01123 & 01123 \\ \hline 0 & 10111 & 10111 & 10111 \end{array} \right]$$

generates the code

$$M_{3,2}(4) = \{(0ccc) + \alpha(1123) \mid \alpha \in \mathbb{Z}_4\}$$

By Theorem 3.1, the weight distribution of $S_2(4)$ is

$$A_0 = 1, A_4 = 11, A_3 = 2, A_5 = 2$$

and hence the $n(0)$ s are such that 5, 1, 2 and 0 respectively.

Case (i). If $\alpha = 0$, then using Equation 3.1, we have

$$wt((0ccc) + \alpha(1123)) = 3wt(c),$$

for all $c \in S_2(4)$. Therefore, by Equation 3.2, there is only one codeword of weight zero, 2 codewords of weight 9, 11 codewords of weight 12 and 2 codewords of weight 15.

Case (ii). If $(\alpha, 4) = 1$, then $\alpha \in \{1, 3\}$, by using Equation 3.3, we have,

$$wt((0ccc) + \alpha(1123)) = 1 + (2)(5) + n(0) = 11 + n(0)$$

By Equation 3.4, there are 2 codewords of weight 16, 4 codewords of weight 13, 22 codewords of weight 12 and 4 codewords of weight 11.

Case (iii). If α is not relatively prime to 4, then $\alpha \in \{2\}$ and by Equation 3.5, we get,

$$\begin{aligned} wt((0ccc) + 2(1123)) &= 1 + (4-1)(5) - \frac{4}{2} \sum_{i=0}^1 n(i2) + n(0) \\ &= 1 + 15 - 2[n(0) + n(2)] + n(0) \\ wt((0ccc) + 2(1123)) &= 16 - n(0) - 2n(2). \end{aligned}$$

Using the coordinates of $c \in S_2(q)$, there is only one codeword of weight 11, only one codeword of weight 7, 2 codewords of weight 8, 8 codewords of weight 13, 2 codewords of weight 14 and 2 codewords of weight 15.

By combining cases (i), (ii) and (iii), we have

Theorem 3.3. *The weight distribution of \mathbb{Z}_4 -MacDonald code $M_{3,2}(4)$ is*

$$\begin{aligned} A_0 = 1, \quad A_7 = 1, \quad A_8 = 2, \quad A_9 = 2, \quad A_{11} = 5, \quad A_{12} = 33, \quad A_{13} = 12, \\ A_{14} = 2, \quad A_{15} = 4, \quad A_{16} = 2. \end{aligned}$$

4 Weight Distribution of \mathbb{Z}_q -Simplex Code of dimension 3, for any $q \geq 2$.

Let

$$G_3(q) = \left[\begin{array}{cccc|c|cccc|c|c|cccc|c} 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 2 & 2 & 2 & \cdots & 2 & \cdots & q-1 & q-1 & q-1 & \cdots & q-1 \\ \hline 0 & 1 & 1 & \cdots & q-1 & 0 & 0 & 1 & 1 & \cdots & q-1 & 0 & 1 & 1 & \cdots & q-1 & \cdots & 0 & 1 & 1 & \cdots & q-1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 1 & 0 & 1 & \cdots & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \end{array} \right]$$

Then this matrix generates the code $S_3(q) = \{(c0c \cdots c) + \alpha(0112 \cdots \mathbf{q}-1) \mid \alpha \in \mathbb{Z}_q\}$.

In [11], we have given the parameters of $S_k(q)$, and the weight distribution of $S_2(q)$ is given by Theorem 3.1.

Case (i). Let $\alpha = 0$. Then,

$$\begin{aligned} wt((c0cc \cdots c) + \alpha(0112 \cdots \mathbf{q}-1)) &= wt(c0cc \cdots c) \\ &= (q)wt(c), \text{ for all } c \in S_2(q) \end{aligned} \quad (4.1)$$

In this way, we get the following weights.

1. Number of zero weight codeword is 1.
2. Number of $q(q - \frac{q}{d} + 1)$ weight codeword is $d\phi(d)$, where $d|q$, $d \neq 1$ and $d \neq q$.
3. Number of $qq = q^2$ weight codeword is $q\phi(q) + q - 1$.
4. Number of $q(q + 1)$ weight codeword is $q(q - 1) - \sum_{d|q, d \neq 1} d\phi(d)$.

Case (ii). Let $(\alpha, q) = 1$. Since $\{\alpha.1, \alpha.2, \dots, \alpha.(q-1)\} = \{1, 2, \dots, q-1\}$,

$$\begin{aligned}
 wt((c0cc \cdots c) + \alpha(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})) &= wt((c0cc \cdots c) + (\mathbf{0}\alpha\mathbf{12} \cdots \mathbf{q} - \mathbf{1})) \\
 &= 1 + \sum_{i=0}^{q-1} wt(c + \mathbf{i}) \\
 &= 1 + \sum_{i=0}^{q-1} wt(-c + \mathbf{i}) \\
 &= 1 + \sum_{i=0}^{q-1} [n - n(\mathbf{i})] \\
 &= 1 + qn - n \\
 wt((c0cc \cdots c) + \alpha(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})) &= 1 + (q-1)n \text{ for all } c \in S_2(q). \tag{4.2}
 \end{aligned}$$

From the above, the number of $1 + (q-1)n$ weight codeword is $\#(S_2(q)) = q^2$ for all $c \in S_2(q)$.

Since there are $\phi(q)$ α 's such that $(\alpha, q) = 1$, it implies that the number of $1 + (q-1)n$ weight codeword is $\phi(q).q^2$ and hence

$$A_{1+(q-1)n} = \phi(q).q^2 \tag{4.3}$$

Case (iii). If α is not relatively prime to q , then

$$\begin{aligned}
 wt((c0cc \cdots c) + \alpha(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})) &= 1 + wt(c + \alpha\mathbf{0}) + wt(c + \alpha\mathbf{1}) + \cdots + wt(c + \alpha(\mathbf{q} - \mathbf{1})) \\
 &= 1 + wt(0\alpha - c) + wt(1\alpha - c) + \cdots + wt((d-1)\alpha - c) + \cdots \\
 &= 1 + \frac{q}{d} \sum_{i=0}^{d-1} wt(\mathbf{i}\alpha - c) \\
 &= 1 + \frac{q}{d} \sum_{i=0}^{d-1} [n - n(\mathbf{i}\alpha)] \\
 &= 1 + \frac{q}{d} [dn] - \frac{q}{d} \sum_{i=0}^{d-1} n(\mathbf{i}\alpha) \\
 &= 1 + qn - \frac{q}{d} \sum_{i=0}^{d-1} n(\mathbf{i}\alpha)
 \end{aligned}$$

Therefore,

$$wt((c0cc \cdots c) + \alpha(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})) = 1 + qn - \frac{q}{d} \sum_{i=0}^{d-1} n(\mathbf{i}\alpha), \quad (4.4)$$

where $n(\mathbf{i}\alpha)$ is the number of $\mathbf{i}\alpha$'s in c . If we know the details of coordinates in c , we can get the remaining weights of $S_3(q)$.

Example 4.1. For $q = 4, k = 3$, the Matrix

$$G_3(4) = \left[\begin{array}{c|c|c|c|c} 00000 & 1 & 11111 & 22222 & 33333 \\ \hline 01123 & 0 & 01123 & 01123 & 01123 \\ \hline 10111 & 0 & 10111 & 10111 & 10111 \end{array} \right]$$

generates the code

$$S_3(4) = \{(c0ccc) + \alpha(\mathbf{01123}) \mid \alpha \in \mathbb{Z}_4, c \in S_2(4)\} \text{ where } \mathbf{i} = ii \cdots i \in \mathbb{Z}_q^n$$

Case (i). Let $\alpha = 0$. Then, using Equation 4.1, we have

$$wt((c0ccc) + \alpha(\mathbf{01123})) = wt(c0ccc) = 4wt(c),$$

for all $c \in S_2(4)$. Therefore, by using the weight distribution of $S_2(4)$, there is only one codeword of weight zero, 11 codewords of weight 16, 2 codewords of weight 12 and 2 codewords of weight 20.

Case (ii). If $(\alpha, 4) = 1$, then $\alpha \in \{1, 3\}$ and by Equation 4.2, we get

$$wt((c0ccc) + \alpha(\mathbf{01123})) = 1 + 3n$$

Then, by Equation 4.3, the number of $1 + 3n = 16$ weight codeword is $\phi(4).4^2 = 32$. That is, there are 32 codewords of weight 16.

Case (iii). If α is not relatively prime to 4, then $\alpha \in \{2\}$ and by Equation 4.4, we get

$$\begin{aligned} wt((c0ccc) + 2(\mathbf{01123})) &= 1 + (4)(5) - \frac{4}{2} \sum_{i=0}^1 n(i2) \\ &= 21 - 2[n(0) + n(2)]. \end{aligned}$$

Using the coordinates of $c \in S_2(4)$, we get, there are 4 codewords of weight 11, 8 codewords of weight 17 and 4 codewords of weight 19. Therefore, by combining cases (i), (ii) and (iii), we have

Theorem 4.2. The weight distribution of \mathbb{Z}_4 -Simplex code of dimension 3 is

$$A_0 = 1, A_{11} = 4, A_{12} = 2, A_{16} = 43, A_{17} = 8, A_{19} = 4, A_{20} = 2.$$

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